

Axial Gauged Noncommutative $U(2)/U(1)$ Wess-Zumino-Witten Model ¹

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Abstract

We construct various kinds of gauged noncommutative WZW models. In particular, axial gauged noncommutative $U(2)/U(1)$ WZW model is studied and by integrating out the gauge fields, we obtain a noncommutative non-linear σ -model.

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1 Introduction

Noncommutative field theory has emerged from string theory in certain backgrounds [1, 2, 3, 4]. The noncommutativity of space is defined by the relation,

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \quad (1.1)$$

where $\theta^{\mu\nu}$ is a second rank antisymmetric real constant tensor. The function algebra in the noncommutative space is defined by the noncommutative and associative Moyal \star -product,

$$(f \star g)(x) = e^{\frac{i}{2}\theta^{\mu\nu} \frac{\partial}{\partial \xi^\mu} \frac{\partial}{\partial \eta^\nu}} f(\xi)g(\eta)|_{\xi=\eta=x}. \quad (1.2)$$

A noncommutative field theory is simply obtained by replacing ordinary multiplication of functions by the Moyal \star -product. An interesting field theory whose noncommutative version would be of interest is the WZW model. In [5], a noncommutative non-linear σ -model has been studied and an infinite dimensional symmetry is found. They also derived some properties of noncommutative WZW model. In [6], the β -function of the $U(N)$ noncommutative WZW model was calculated and found to be the same as that of ordinary commutative WZW model. Hence, the conformal symmetry in certain fixed points is recovered. In [7] and [8], the derivation of noncommutative WZW action from a gauge theory was carried out. The connection between noncommutative two-dimensional fermion models and noncommutative WZW models was studied in [9, 10].

In this letter, we study the gauged noncommutative WZW models. In section 2, after a brief review of noncommutative WZW model, we construct different versions of gauged noncommutative WZW models. In section 3, we consider the axial gauged noncommutative $U(2)$ WZW model by its diagonal $U(1)$ subgroup. The obtained gauged action contains infinite derivatives in its \star structure and hence is a nonlocal field theory. Integration over the gauge fields requires solving an integral equation which we solve by perturbative expansion in θ . The result is a noncommutative non-linear σ -model, which may contains singular structures or a black hole.

2 Different Versions of Gauged Noncommutative WZW Models

The action of the noncommutative WZW model is [7]:

$$S(g) = \frac{k}{4\pi} \int_{\Sigma} d^2z \operatorname{Tr}(g^{-1} \star \partial g \star g^{-1} \star \bar{\partial} g) - \frac{k}{12\pi} \int_M \operatorname{Tr}(g^{-1} \star dg)^3_{\star}, \quad (2.1)$$

where M is a three-dimensional manifold whose boundary is Σ , and g is a map from Σ (or from its extension M) to the group G . We assume that the coordinates (z, \bar{z}) on the worldsheet Σ are noncommutative but the extended coordinate t on the manifold M commutes with others:

$$[z, \bar{z}] = \theta, \quad [t, z] = [t, \bar{z}] = 0. \quad (2.2)$$

We define the group-valued field g by,

$$g = e^{i\pi^a T_a}_{\star} = 1 + i\pi^a T_a + \frac{1}{2}(i\pi^a T_a)^2_{\star} + \cdots, \quad (2.3)$$

where the T_a 's are the generators of group G .

Inserting the \star -product of two group elements in the eq. (2.1), we find the noncommutative Polyakov-Wiegmann identity,

$$S(g \star h) = S(g) + S(h) + \frac{1}{16\pi} \int d^2z \operatorname{Tr}(g^{-1} \star \bar{\partial} g \star \partial h \star h^{-1}), \quad (2.4)$$

which is the same as ordinary commutative identity with products replaced by \star -products.

Using the Polyakov-Wiegmann identity, we can show that the action (2.1) is invariant under the following transformations:

$$g(z, \bar{z}) \rightarrow h(z) \star g(z, \bar{z}) \star \bar{h}(\bar{z}). \quad (2.5)$$

The corresponding conserved currents are,

$$\begin{aligned} J(z) &= \frac{k}{2\pi} \bar{\partial} g \star g^{-1}, \\ \bar{J}(\bar{z}) &= \frac{k}{2\pi} g^{-1} \star \partial g. \end{aligned} \quad (2.6)$$

By use of the equations of motion, we can show that these currents are indeed conserved,

$$\bar{\partial} J(z) = \partial \bar{J}(\bar{z}) = 0. \quad (2.7)$$

The quantization of the noncommutative WZW model was done in [12], and the current algebra of the noncommutative WZW model found to be,

$$[J_a(\sigma), J_b(\sigma')] = \delta(\sigma - \sigma') if_{ab}^c J_c(\sigma) + \frac{k}{2\pi} i \delta'(\sigma - \sigma') \delta_{ab}, \quad (2.8)$$

and a similar relation for commutation of \bar{J} 's.

Note that in the above commutation relation, θ does not appear, and it is just as commutative ordinary affine algebra with the same central charge. The absence of θ has been expected, since the currents are holomorphic by equations of motion and hence commutative in the sense of \star -product.

Constructing the energy momentum tensor is also straightforward,

$$T(z) = \frac{1}{k+N} : J_i(z) J_i(z) : + \frac{1}{k} : J_0(z) J_0(z) :, \quad (2.9)$$

where J_i 's are $SU(N)$ currents and J_0 is the $U(1)$ current corresponding to the subgroups of $U(N) = U(1) \times SU(N)$. Again the products in (2.9) are commutative because of holomorphicity of the currents. So the Virasoro algebra is also the same as usual standard form and its central charge is unchanged,

$$c = \frac{kN^2 + N}{k+N}. \quad (2.10)$$

We want to gauge the chiral symmetry (2.5) as,

$$g(z, \bar{z}) \rightarrow h_L(z, \bar{z}) \star g(z, \bar{z}) \star h_R(z, \bar{z}), \quad (2.11)$$

where h_L and h_R belong to H some subgroup of G . For finding the invariant action under the above transformation we need to add gauge fields terms to the action (2.1) as follows:

$$S(g, A, \bar{A}) = S(g) + S_A + S_{\bar{A}} + S_2 + S_4, \quad (2.12)$$

where, $S(g)$ is the action (2.1) and

$$\begin{aligned} S_A &= \frac{k}{4\pi} \int d^2z \text{Tr}(A_L \star \bar{\partial}g \star g^{-1}), \\ S_{\bar{A}} &= \frac{k}{4\pi} \int d^2z \text{Tr}(\bar{A}_R \star g^{-1} \star \partial g), \\ S_2 &= \frac{k}{4\pi} \int d^2z \text{Tr}(\bar{A}_R \star A_L), \\ S_4 &= \frac{k}{4\pi} \int d^2z \text{Tr}(\bar{A}_R \star g^{-1} \star A_L \star g). \end{aligned} \quad (2.13)$$

Gauge transformations for the gauge fields are

$$\begin{aligned} A_L &\rightarrow h_L \star (A_L + d) \star h_L^{-1}, \\ A_R &\rightarrow h_R^{-1} \star (A_R + d) \star h_R. \end{aligned} \quad (2.14)$$

Using the Polyakov-Wiegmann identity (2.4), one can find the transformed form of $S(g)$ and the gauge field terms (2.13), under the transformations (2.11) and (2.14) [12].

To find an invariant action $S(g, A, \bar{A})$, we have to choose constraints on the subgroup elements h_L and h_R . The first consistent choice is, $h_R = h_L^{-1} \equiv h$, and yields to following transformations,

$$\begin{aligned} g &\rightarrow g' = h^{-1} \star g \star h, \\ A &\rightarrow A' = h^{-1} \star (A \star h + \partial h), \\ \bar{A} &\rightarrow \bar{A}' = h^{-1} \star (\bar{A} \star h + \bar{\partial} h). \end{aligned} \quad (2.15)$$

The corresponding invariant action, called vector gauged WZW action, is,

$$S_V(g, A, \bar{A}) = S(g) + S_A - S_{\bar{A}} + S_2 - S_4. \quad (2.16)$$

The second choice is to take $h_L = h_R \equiv h$ with h belonging to an Abelian subgroup of G . In this case we find the following gauge transformations,

$$\begin{aligned} g &\rightarrow g' = h \star g \star h, \\ A &\rightarrow A' = h \star (A \star h^{-1} + \partial h^{-1}), \\ \bar{A} &\rightarrow \bar{A}' = h^{-1} \star (\bar{A} \star h - \bar{\partial} h), \end{aligned} \quad (2.17)$$

with the so called axial gauged WZW action,

$$S_A(g, A, \bar{A}) = S(g) + S_A + S_{\bar{A}} + S_2 + S_4. \quad (2.18)$$

By integrating out the A and \bar{A} from the actions (2.16) and (2.18), in principle, we find the effective actions as noncommutative non-linear σ -models.

3 Axial Gauged Noncommutative $U(2)/U(1)$ WZW Model

We take here the noncommutative gauged axial $U(2)$ WZW action (2.18), gauged by the subgroup $U(1)$ diagonally embedded in $U(2)$. The group element of $U(2)$ is

$$g = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, \quad (3.1)$$

with the following constraints,

$$\begin{aligned} a_1 \star a_1^\dagger + a_2 \star a_2^\dagger &= 1, \\ a_3 \star a_3^\dagger + a_4 \star a_4^\dagger &= 1, \\ a_1 \star a_3^\dagger + a_2 \star a_4^\dagger &= 0. \end{aligned} \quad (3.2)$$

The gauge parts of the action (2.18) is

$$S_{gauge} = \frac{k}{2\pi} \int d^2z \{ A \sum_i \bar{\partial} a_i \star a_i^\dagger + \sum_i a_i^\dagger \star \partial a_i \bar{A} + 2A\bar{A} + \sum_i A \star a_i \star \bar{A} \star a_i^\dagger \}. \quad (3.3)$$

To illustrate the integrating over the gauge fields A and \bar{A} , we consider abbreviated notations as follows:

$$\begin{aligned} \int \mathcal{D}A \mathcal{D}\bar{A} e^{-S_{gauge}} &= \int \mathcal{D}A \mathcal{D}\bar{A} e^{-\int d^2z (A \star \mathcal{O} \star \bar{A} + b \star \bar{A} + A \star \bar{b})} \\ &= \int \mathcal{D}A \mathcal{D}\bar{A} e^{-\int d^2z ((A+b') \star \mathcal{O} \star (\bar{A}+\bar{b}') - b' \star \mathcal{O} \star \bar{b}')}, \end{aligned} \quad (3.4)$$

where

$$b' \star \mathcal{O} = b, \quad \mathcal{O} \star \bar{b}' = \bar{b}. \quad (3.5)$$

The result of integration would be

$$e^{-S_{eff}} = (\det \mathcal{O})^{-1/2} e^{\int d^2z b' \star \bar{b}} e^{-S(g)}. \quad (3.6)$$

By comparing eq. (3.4) with (3.3), b and \bar{b} could be read as follows:

$$b = \frac{k}{2\pi} \sum_i a_i^\dagger \star \partial a_i, \quad \bar{b} = \frac{k}{2\pi} \sum_i \bar{\partial} a_i \star a_i^\dagger, \quad (3.7)$$

and \mathcal{O} can be read from quadratic terms of gauge fields A and \bar{A} in (3.3). In fact by using the Fourier transformation of Moyal \star -products of functions, the explicit Fourier transform of \mathcal{O} is as follows:

$$\begin{aligned} \mathcal{O}(p_1, p_2) &= 2e^{i(p_1 \wedge p_2)} \delta(p_1 + p_2) + \int a_i(p_3) a_i^\dagger(p_4) e^{i(p_1 \wedge p_2 + p_1 \wedge p_3 + p_1 \wedge p_4 + p_2 \wedge p_4 + p_3 \wedge p_4 + p_3 \wedge p_2)} \\ &\quad \times \delta(p_1 + p_2 + p_3 + p_4) dp_3 dp_4. \end{aligned} \quad (3.8)$$

To find b' , we need to inverse the \mathcal{O} operator in (3.5), and this is equivalent to solve Fourier transform of (3.5) which is an integral equation as follows:

$$2b'(p) + \int b'(p - p_1 - p_2) a_i(p_1) a_i^\dagger(p_2) e^{-i(p \wedge p_2 + p_1 \wedge p + p_2 \wedge p_1)} dp_1 dp_2 = b(p), \quad (3.9)$$

where

$$b(p) = i \int a_i^\dagger(p - p_1) a_i(p_1) p_1 e^{ip \wedge p_1} dp_1. \quad (3.10)$$

To solve eq. (3.9), we expand the b' and exponential factors in terms of θ , $b'(p) = b'_0(p) + \theta b'_1(p) + \theta^2 b'_2(p) + \dots$.

One finds [12],

$$b'_0(z, \bar{z}) = \frac{1}{4} a_i^\dagger \partial a_i, \quad b'_1(z, \bar{z}) = \partial \partial a_i \bar{\partial} a_i^\dagger - \bar{\partial} \partial a_i \partial a_i^\dagger. \quad (3.11)$$

In obtaining the above expressions, we have used the unitarity conditions (3.2) and the equations of motion (2.7). The effective action arising from the $\int d^2 z b' \star \bar{b}$ term in eq. (3.6) could be found as a power series in θ ,

$$S_{eff} = S(g) + \frac{1}{2} Tr \ln(\mathcal{O}) + S_{eff}^{(0)} + \theta S_{eff}^{(1)} + \dots, \quad (3.12)$$

in which ³

$$\begin{aligned} S_{eff}^{(0)} &= -\frac{k}{8\pi} \int d^2 z a_i^\dagger \partial a_i \bar{\partial} a_j a_j^\dagger, \\ S_{eff}^{(1)} &= -\frac{k}{8\pi} \int d^2 z ((\partial a_i^\dagger \partial a_i + a_i^\dagger \partial \partial a_i)(\bar{\partial} \bar{\partial} a_j a_j^\dagger + \bar{\partial} a_j \bar{\partial} a_j^\dagger) \\ &\quad + a_i^\dagger \partial a_i (\partial \bar{\partial} a_j \bar{\partial} a_j^\dagger - \bar{\partial} \bar{\partial} a_j \partial a_j^\dagger) - (\bar{\partial} \partial a_i \partial a_i^\dagger - \partial \partial a_i \bar{\partial} a_i^\dagger) \bar{\partial} a_k a_k^\dagger). \end{aligned} \quad (3.13)$$

By looking at equations (3.11) and (3.13), we suggest the following exact forms for $b'(z, \bar{z})$ and S_{eff} ,

$$\begin{aligned} b'(z, \bar{z}) &= \frac{1}{4} \partial a_i \star a_i^\dagger, \\ S_{eff} &= S(g) + \frac{1}{2} Tr \ln(\mathcal{O}) - \frac{k}{8\pi} \int d^2 z \partial a_i \star a_i^\dagger \star \bar{\partial} a_j \star a_j^\dagger. \end{aligned} \quad (3.14)$$

We have to fix the gauge freedom by a gauge fixing condition on a_i 's. Under infinitesimal axial gauge transformations (2.17), we find, $a'_i = a_i + a_i \star \epsilon + \epsilon \star a_i$, in which ϵ is the infinitesimal parameter of the gauge transformation. We can find (at least perturbatively in θ) some ϵ such that $\Re(a'_1) = 0$ and one may take this relation as the gauge fixing condition. It is worth mentioning that the $U(2)/U(1)$ model after applying all conditions gives us a three dimensional non-linear noncommutative σ -model. The geometrical study of this target space which may contains singular structure will be interesting [13].

³The term $\frac{1}{2} Tr \ln(\mathcal{O})$ gives the noncommutative effective action for dilaton field.

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